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A NOTE ON EUCLIDEAN RAMSEY THEORY AND A CONSTRUCTION OF BOURGAIN

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1. Qualitative facts

Let v be a fixed unit vector in a Hilbert space Ω. Denote

$$\Omega_c = \{\omega\in\Omega \mid \langle v, \omega\rangle = c,\ \|\omega\| = 1\}$$

for a real $0<c<1$. Bessel's inequality implies that any orthogonal sequence in Ω_c is finite. Thus, Ramsey's theorem implies

FACT 1. *From any infinite sequence $\{\omega_n\}_{n=1}^{\infty}$ in Ω_c an infinite subsequence can be extracted, with no two vectors orthogonal.*

We will be interested in the "size" of the subsequence which can be extracted, especially when a further restriction is put on the sequence $\{\omega_n\}$. In particular, we show that a subsequence of positive density cannot always be extracted.

DEFINITIONS. I. A sequence of vectors $\{\omega_n\}$ in a Hilbert space is *stationary* if $\langle\omega_{i+n}, \omega_{j+n}\rangle = \langle\omega_i, \omega_j\rangle$ for all i, j, n.

II. A set of integers $H\subset\mathbf{N}$ is a *Van der Corput set* if every probability measure μ on the circle satisfying $\hat\mu(h) = \int e^{-iht}\,d\mu(t) = 0$ for every $h\in H$ satisfies $\mu\{0\} = 0$.

III. A set of integers $H\subset\mathbf{N}$ is a *Poincare set* if for every set $S\subset\mathbf{N}$ of positive density, H intersects the difference set $S-S$. (For an alternative ergodic theory definition see [3].)

Kamae and Mendes France [5] proved that all Van der Corput sets are Poincare sets. Recently, J. Bourgain [1] has proved that the reverse implication does not hold. This implies

FACT 2. *There exist a $0<c<1$ and a stationary sequence of vectors $\{\omega_n\}$ in Ω_c such that for any $S\subset\mathbf{N}$ of positive density, $\omega_n\perp\omega_m$ for some $m, n\in S$.*

PROOF Let H be a Poincare set which is not Van der Corput. There exists a measure μ for which $\hat\mu(n) = 0$ $\forall n\in H$ and $\mu\{0\} = c^2 > 0$. Let Ω be the Hilbert space $L^2[0, 2\pi)$. Let $\omega_n(t) = e^{int}$, and denote

$$v(t) = \begin{cases} c^{-1}, & t = 0 \\ 0, & t\neq 0.\end{cases}$$

Clearly $\omega_n\in\Omega_c$. For any sequence $S\subset\mathbf{N}$ of positive density, some $m, n\in S$ satisfy $m-n\in H$ and hence $\hat\mu(m-n) = \langle\omega_m, \omega_n\rangle = 0$. □

Bourgain's construction is difficult; thus we note

FACT 3. *From any sequence* $\{\omega_n\}$ *satisfying the conclusion of Fact* 2, *one can easily construct a Poincare set which is not Van der Corput.*

PROOF. By the stationarity of $\{\omega_n\}$, the sequence $\{\langle\omega_n, \omega_0\rangle\}$ is positive definite, so by Herglotz's theorem [6], there exists a positive measure μ on the circle, such that $\hat{\mu}(n)=\langle\omega_n, \omega_0\rangle$ for all n. From $\langle\omega_n, v\rangle=c>0$ it easily follows that $\mu\{0\}>0$. (Indeed, $\{\omega_n-cv\}$ is stationary and hence there is a positive measure ν so that $\hat{\nu}(n)=\langle\omega_n-cv, \omega_0-cv\rangle=\hat{\mu}(n)-c^2$. This implies $\mu=\nu+c^2\delta_0$ and $\mu\{0\}\geqq c^2$.) Thus $H=\{n>0|\hat{\mu}(n)=0\}$ is the desired Poincare set. □

If we ignore the geometry and concentrate on the combinatorics of Fact 2, we get

FACT 4. *For some* K_0, *the edges of the complete graph on* $\mathbf{N}$ *can be 2-coloured so that*

I. *there is no white Clique of size* K_0,
II. *there is no black Clique of positive upper density, and*
III. *the colouring is stationary:* $\{i, j\}$ *and* $\{i+n, j+n\}$ *are coloured identically.*

H. Furstenberg and B. Weiss [private communication] have given an elegant example which shows Fact 4 with $K_0=3$: Colour $\{i, j\}$ white if for some integer x, $i-j=x^3$, black otherwise. There is no white clique of size 3, because of Fermat's last theorem with exponent 3; there is no black clique of positive density because the set $\{x^3\}_{x\in\mathbf{N}}$ is a Poincare set (see [3]). □

2. Two Ramsey-like functions

DEFINITION. For $0<c<1$, define a function Λ_c: $\mathbf{N}\to\mathbf{N}$ as follows: $\Lambda_c(k)$ is the minimal N such that from any *stationary* sequence $\{\omega_n|0\leqq n<N\}$ in Ω_c, k elements can be extracted, no two of which are orthogonal. $\Gamma_c(k)$ is defined similarly, *without* the stationarity constraint.

Clearly $\Lambda_c\leqq\Gamma_c$.

FACT 5. $\Gamma_c(2)=\Lambda_c(2)=\lfloor c^{-2}\rfloor+1$.

PROOF. Put $N=N_c=\lfloor c^{-2}\rfloor+1$ and $d=\sqrt{1-(N-1)c^2}$. Let A be an orthogonal N by N matrix whose first column is the vector $(c, c, \ldots, c, d)$. Let v be the N-dimensional vector $(1, 0, \ldots, 0)$, and let $\omega_0, \ldots, \omega_{N-2}$ be the first $N-1$ row vectors of A. Clearly $\omega_n\in\Omega_c$ and $\langle\omega_n, \omega_m\rangle=0$. Thus $\Gamma_c(2)\geqq\Lambda_c(2)>N-1$. It remains to show that $\Gamma_c(2)\leqq N$. Indeed, if this is false, there are N orthogonal vectors $\{\omega_n|0\leqq n<N\}$ in Ω_c. Bessels inequality $\|v\|^2\geqq\sum|\langle v, \omega_i\rangle|^2=c^2\cdot N>1$ gives the desired contradiction. □

FACT 6. $\Gamma_c(k)\leqq R(N_c, k)$ *where* $R(N, k)\leqq\binom{N+k-2}{N-1}$ *is the Ramsey number corresponding to* N *and* k (see [4]).

This is immediate from Fact 5. □

The upper bound above is not tight. For Λ_c we do not have a better upper bound. Regarding lower bounds we note

PROPOSITION 1. *$\Lambda_c(k)$ does not increase linearly with k, for some $0<c<1$.*

PROPOSITION 2. *There exist $0<c<1$, $\alpha>1$ and an increasing sequence $\{k_l \mid l \geqq 1\}$ satisfying $\Gamma_c(k_l) \geqq k_l^\alpha$ for all l.*

Proposition 1 follows from Fact 2; Proposition 2 is a consequence of the following result, due to Frankl and Wilson [2]:

THEOREM. [2]. *Let $\mathcal{F}$ be a family of subsets of $\{1, \ldots, n\}$ such that for every $F \in \mathcal{F}$, $|F|=k$, and let $q<k$ be a prime power. If every different $F, F' \in \mathcal{F}$ satisfy $|F \cap F'| \not\equiv$ $\not\equiv k \bmod q$ then $|\mathcal{F}| \leqq \binom{n}{q-1}$.*

PROOF. Denote $n=2^l$, $N=\binom{n}{\frac{3n}{8}}$ and let $\{F_j\}_{j=1}^N$ be all subsets of $\{1, \ldots, n\}$ of size $\frac{3n}{8}$. Define vectors $\{\omega_i\}_{i=1}^N$ in $\mathbf{R}^n$ by

$$\omega_i = n^{-1/2}(2 \cdot 1_{F_i} - 1)$$

where 1_F is the indicator vector of F.

Define also $v = -n^{-1/2}(1, 1, \ldots, 1) \in \mathbf{R}^n$. For $1 \leqq i \leqq N$ we get

$$\|v\| = \|\omega_j\| = 1, \quad \langle v, \omega_i \rangle = \frac{1}{4} = c,$$

$$\omega_i \perp \omega_j \Leftrightarrow |F_i \cap F_j| = n/8 \equiv \frac{3n}{8} \left(\bmod \frac{n}{4}\right).$$

$q=\frac{n}{4}$ is a power of 2. Thus the theorem cited above shows that any subset $\mathcal{F}$ of $\{\omega_1, \ldots, \omega_N\}$ which does not contain orthogonal vectors, satisfies

$$|\mathcal{F}| \leqq \binom{n}{\frac{n}{4}-1}.$$

In other words, for $k_l = \binom{n}{n/4}$, $\Gamma_c(k_l) > \binom{3}{\frac{3n}{8}}$ and

$$\lim_{l \to \infty} \frac{\log \Gamma_c(k_l)}{\log k_l} \geqq \frac{h\left(\frac{3}{8}\right)}{h\left(\frac{1}{4}\right)} > 1$$

where $h(x) = -x \log x - (1-x) \log (1-x)$ is the binary entropy function. Any α smaller than the entropy ratio above will do. □

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